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A pre-quantum classical statistical model with infinite-dimensional phase space

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Abstract

We study the problem of correspondence between classical and quantum statistical models. We show that (contrary to a rather common opinion) it is possible to construct a natural pre-quantum classical statistical model. The crucial point is that such a pre-quantum classical statistical model is not the conventional classical statistical mechanics on the phase space $\mathbf{R}^{2n}$, but its infinite-dimensional analogue. Here the phase space $\Omega = H \times H$, where H is the (real separable) Hilbert space. The classical $\rightarrow$ quantum correspondence is based on the Taylor expansion of classical physical variables—maps $f : \Omega \rightarrow \mathbf{R}$. The space of classical statistical states consists of Gaussian measures on Ω having zero mean value and dispersion $\approx h$. The quantum statistical model is obtained as the $\lim_{h\to 0}$ of the classical one. All quantum states including so-called 'pure states' (wavefunctions) are simply Gaussian fluctuations of the 'vacuum field', $\omega = 0 \in \Omega$, having dispersions of the Planck magnitude.

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1. Introduction

Since the first days of the creation of quantum mechanics, physicists, mathematicians and philosophers are involved in stormy debates on the possibility of creating a classical pre-quantum statistical model; see, for example, [1–44]. Here 'classical statistical' has the meaning of a realistic model, in which physical variables can be considered as objective properties and probabilities can be described by the classical (Kolmogorov) measure-theoretic model. There is a rather common opinion that it is impossible to construct such a pre-quantum model. Such an opinion is a consequence of Bohr's belief that quantum mechanics is a *complete theory.* Therefore it is, in principle, impossible to create a deeper description of physical reality. In particular, there is a rather common belief that quantum randomness is irreducible; see, e.g., [4] (in contrast to classical randomness which is reducible in the sense that it can be reduced to ensemble randomness of objective properties). There is much activity in proving various mathematical 'no-go' theorems (e.g., von Neumann[1], Kochen–Specker, Bell, and others). Many people think that with the aid of such mathematical investigations it is possible to prove completeness of quantum mechanics. As was pointed out in the preface to the conference proceedings [36], such an approach cannot be justified, because we do not know the *correspondence rules* between pre-quantum and quantum models.

J von Neumann presented in his book [4] a list of possible features of such a classical $\rightarrow$ quantum map T. Later this list was strongly criticized by many authors (including J Bell) [12]. In particular, there was criticism on the assumption of one-to-one correspondence between the set of classical physical variables V and the set of quantum observables O. There it was also pointed out that von Neumann's assumption that $T(a + b) = T(a) + T(b)$ for any two physical variables (without the assumption that observables $T(a)$ and $T(b)$ can be measured simultaneously) is non-physical. Then different authors proposed their own lists of possible features of the map T which (as they think) are natural. These lists (including Bell's list) were again criticized; see, e.g., [26, 27–29, 37, 40, 41, 44] and some papers in [33–36].

In [45] I proposed to start the activity in the opposite direction. Instead of looking for lists of assumptions on the classical $\rightarrow$ quantum map T which would imply a new 'no-go' theorem, it seems more natural to try to find such lists of features of T which would give the possibility of creating a natural pre-quantum classical statistical model. In these papers, it was shown that all distinguishing features of the quantum probabilistic model (interference of probabilities, Born's rule, complex probabilistic amplitudes, Hilbert state space, representation of observables by operators) are present in a latent form in the classical Kolmogorov probability model.

The approach developed in [45] has a few weak sides. A pure mathematical problem is that this approach was developed only for discrete observables. We also pay attention to two fundamental physical problems which are in fact closely coupled: (a) the framework of [45] does not give the possibility of selecting the 'right' (from the physical point of view) pre-quantum model among all possible (from the mathematical point of view) Kolmogorov models, because in papers [45] QM is considered as purely mathematical probabilistic formalism which is characterized by contextuality of probabilities (so there was nothing new about physical laws, but just about mathematical laws for transformations of probabilities depending on contexts); (b) the role of the Planck constant h in classical $\rightarrow$ quantum correspondence was not clear; see [46] for debates[2].

In this paper, we construct a physically adequate pre-quantum classical statistical model. The crucial point is that the pre-quantum classical statistical model is not the conventional classical statistical mechanics on the phase space $\mathbf{R}^{2n}$, but its infinite-dimensional analogue. Here the phase space $\Omega = H \times H$, where H is the (real separable) Hilbert space. The classical $\rightarrow$ quantum correspondence is based on the Taylor expansion of classical physical variables—maps $f : \Omega \rightarrow \mathbf{R}$. The space of classical statistical states consists of Gaussian measures on Ω having zero mean value and dispersion $\approx h$. The quantum statistical model is obtained as the $\lim_{h\rightarrow 0}$ of the classical one. All quantum states including so-called 'pure

[1] Recently A Leggett brought to my attention the fact that J von Neumann did not consider his considerations as a rigorous mathematical theorem. In the original German addition (1933) of his book he called 'no-go' considerations ansatz and not theorem.

[2] In the conventional approaches, there is typically considered quantum $\rightarrow$ classical correspondence; see, e.g., [47–50]. In particular, in the formalism of deformation quantization classical mechanics on the phase space $\Omega = \mathbf{R}^{2n}$ is obtained as the $\lim_{h\rightarrow 0}$ of quantum mechanics. Contrary to the conventional approaches, we consider classical $\rightarrow$ quantum correspondence.

states' (wavefunctions) are simply Gaussian fluctuations of the 'vacuum field', $\omega = 0 \in \Omega$, having dispersions of the Planck magnitude.

It is especially interesting that in our approach 'pure quantum states' are not pure at all. These are also statistical mixtures of small Gaussian fluctuations of the 'vacuum field'. It seems that the commonly supported postulate (see, e.g., [4]) about *irreducible quantum randomness*, i.e., randomness which could not be reduced to classical ensemble randomness, was not justified.

We shall discuss a possible physical interpretation of our model and its relation to other realistic pre-quantum models in section 9: the pilot wave model (Bohmian mechanics), see e.g. [7, 11, 19], stochastic QM (in particular, SED), see e.g. [18, 51, 52] and references therein. Briefly, we can say that elementary particles, such as photons or electrons, are not present in our pre-quantum realistic model. Physical reality is reality of 'classical fields'—systems with the infinite number of degrees of freedom (so we consider a field model, but this is not QFT, because quantum mechanics is reproduced not through quantum fields, but classical fields). Images of quantum particles are created in the processes of measurements performed on classical fields. The main distinguishing feature of classical pre-quantum fields is very small magnitudes. Here smallness is understood as *statistical smallness*. We are able to prepare for quantum measurements statistical states (measures on the space of classical fields) having zero mean value and dispersion of the Planck magnitude. In such a statistical ensemble (of classical fields) deviations of fields from the vacuum field are of the magnitude $\sigma(\rho) = \sqrt{h}$. Quantum mechanics is not complete, because our ontic model (describing reality as it is) contains even statistical states describing statistical deviations of the magnitude $\sigma(\rho) = o(\sqrt{h})$. Such statistical states are neglected in the process of classical $\rightarrow$ quantum correspondence. QM does not contain images of these states.

Of course, this paper is just the first step in representing QM as an asymptotic projection of statistical mechanics of classical fields. At the moment, it is too early to discuss possible impact of the presented 'pre-quantum model'. I would not say that the well-known problem of hidden variables is solved. Our pre-quantum model is rather far from original expectations; cf, e.g., with investigations of von Neumann and Bell. The crucial difference is that in conventional models with hidden variables there was assumed coincidence of quantum and classical averages, but in our approach they coincide only asymptotically, so this is an *asymptotic hidden variables model*. Another important problem for further investigations is the study of composite systems and, in particular, an interpretation of Bell's inequality in such a framework. This is the difficult problem and some deviations from QM might be expected.

2. On classical → quantum correspondence

2.1. Ontic and epistemic models

We show that (contrary to the very common opinion) it is possible to construct a pre-quantum classical statistical model. From the very beginning, we should understand that pre-quantum and quantum models give us two different levels of description of physical reality. By using the terminology of Scheibe, Primas and Atmanspacher (see, e.g., [37]) we can say that pre-quantum and quantum models provide, respectively, *ontic* and *epistemic* descriptions. The first describes nature as it is (as it is 'when nobody looks'). The second is an observational model. It gives an image of nature through a special collection of observables.

In any ontic ('realistic') model the following sets are given: (a) Ω—states; (b) $V(\Omega)$—physical variables. Elements of $V(\Omega)$ describe objective properties. In general, it is not assumed that they can be measured. In a statistical ontic model, *statistical states* are also considered; these are distributions of states. Thus an *ontic statistical model* is a couple $M = (S(\Omega), V(\Omega))$, where $S(\Omega)$ is a set of statistical states of a model.

In any epistemic ('observational') statistical model there are given sets of observables and statistical states: O and D. An *epistemic ('observational') statistical model* is a couple $N = (D, O)$. Elements of the set O do not describe objective properties; they describe results of observations. Statistical states represent distributions of states $\omega \in \Omega$.[3] In general ('individual states') ω do not belong to the domain of an epistemic model $N = (D, O)$ (because observers using this model in general are not able to prepare 'individual states' ω). The set of states D of N need not contain images of δ_ω measures concentrated at points $\omega \in \Omega$.

Of course, in physics some epistemic statistical models are used which describe even 'individual states' (belonging to the domain of the corresponding ontic model). Here all measures δ_ω, $\omega \in \Omega$, belong to the set of statistical states D. However, in such a case one need not distinguish ontic and epistemic levels of description.

For example, we can consider *classical statistical mechanics*. Here states are given by points $\omega = (q, p)$ of the phase space $\Omega = \mathbf{R}^{2n}$ and statistical states by probability distributions on Ω. States $\omega \in \Omega$ can be represented by statistical states—δ_ω measures on the phase space.

In the present paper, we are not interested in such statistical models. We are interested in epistemic models which do not provide the description of 'individual states'. In such a case δ_ω measures are not represented by statistical states of an epistemic model: D does not contain $T(\delta_\omega)$, where T is a map performing correspondence between the ontic (pre-observational) model M and the epistemic (observational) model N.

We now discuss mathematical representations of ontic and epistemic models.

2.2. Classical statistical models

Of course, there are many ways to proceed mathematically both on the ontic and epistemic levels of description of nature. But traditionally ontic models are represented as 'classical statistical models':

(a) Physical states ω are represented by points of some set Ω (state space).
(b) Physical variables are represented by functions $f : \Omega \to \mathbf{R}$ belonging to some functional space $V \equiv V(\Omega)$.[4]
(c) Statistical states are represented by probability measures on Ω belonging to some class $S \equiv S(\Omega)$.[5]
(d) The average of a physical variable (which is represented by a function $f \in V(\Omega)$) with respect to a statistical state (which is represented by a probability measure $\rho \in S(\Omega)$) is given by

$$\langle f \rangle_\rho \equiv \int_\Omega f(\omega)\, \mathrm{d}\rho(\omega). \tag{1}$$

A *classical statistical model* is a couple $M = (S, V)$.

We recall that classical statistical mechanics on the phase space $\mathbf{R}^{2n}$ gives an example of a classical statistical model. But we shall not be interested in this example in our further considerations. We shall use a classical statistical model with *an infinite-dimensional phase space*.

[3] Our considerations are based on the realistic approach to the description of physical reality.
[4] The choice of a concrete functional space $V(\Omega)$ depends on various physical and mathematical factors.
[5] It is assumed that there is given a fixed σ field of subsets of Ω denoted by F. Probabilities are defined on F, see [53], 1933. It is, of course, assumed that physical variables are represented by random variables—measurable functions. The choice of a concrete space of probability measures $S(\Omega)$ depends on various physical and mathematical factors.

Remark 2.1. We emphasize that the space of variables $V(\Omega)$ need not coincide with the space of all random variables $RV(\Omega)$—measurable functions $\xi : \Omega \to \mathbf{R}$. For example, if Ω is a differentiable manifold, it is natural to choose $V(\Omega)$ consisting of smooth functions; if Ω is an analytic manifold, it is natural to choose $V(\Omega)$ consisting of analytic functions and so on. Denote the space of all probability measures on the σ field Σ by the symbol $PM(\Omega)$. The space of statistical states $S(\Omega)$ need not coincide with $PM(\Omega)$. For example, for some statistical model $S(\Omega)$ may consist of Gaussian measures.

We shall be interested in ontic models (which are mathematically represented as classical statistical models) inducing the quantum epistemic (observational) statistical model N_{quant}.

2.3. *The quantum statistical model*

In the Dirac–von Neumann formalism [2, 4] in the complex Hilbert space H_c this model is described in the following way:

(a) Physical observables are represented by operators $A : H_c \to H_c$ belonging to the class of continuous[6] self-adjoint operators $\mathcal{L}_{\text{s}} \equiv \mathcal{L}_{\text{s}}(H_c)$ (so O is mathematically represented by $\mathcal{L}_{\text{s}}$).
(b) Statistical states are represented by density operators, see [4]. The class of such operators is denoted by $\mathcal{D} \equiv \mathcal{D}(H_c)$ (so D is mathematically represented by $\mathcal{D}$).
(c) The average of a physical observable (which is represented by the operator $A \in \mathcal{L}_{\text{s}}(H_c)$) with respect to a statistical state (which is represented by the density operator $D \in \mathcal{D}(H_c)$) is given by von Neumann's formula:

$$\langle A \rangle_D \equiv \operatorname{Tr} DA. \tag{2}$$

The *quantum statistical model* is the couple $N_{\text{quant}} = (\mathcal{D}(H_c), \mathcal{L}_{\text{s}}(H_c))$.

Sometimes *pure quantum states* given by normalized vectors $\psi \in H_c$ are also considered. We shall not do this, because 'pure states' of conventional quantum mechanics do not coincide with ontic states of our model. We shall see that pure states are in fact not pure at all. They are statistical mixtures of Gaussian fluctuations of ontic ('individual') states. We just recall that many authors (see, e.g., [28]) define the quantum model in the same way, i.e., without considering pure quantum states.

2.4. *Postulates of classical → quantum correspondence*

As was already pointed out, we are looking for a classical statistical model $M = (S(\Omega), V(\Omega))$ inducing the quantum statistical model $N_{\text{quant}} = (\mathcal{D}(H_c), \mathcal{L}_{\text{s}}(H_c))$. The main problem is that the meaning of the term 'inducing' was not specified![7] For example, one may postulate (see, e.g., [4], p 313) that

Postulate VO. *There is one-to-one correspondence between the space of variables $V(\Omega)$ and the space of observables $\mathcal{L}_{\text{s}}(H_c)$.*

In such a case one could define a one-to-one map:

$$T : V(\Omega) \to \mathcal{L}_{\text{s}}(H_c). \tag{3}$$

[6] To simplify considerations, we shall consider only quantum observables represented by bounded operators. To obtain the general quantum model with observables represented by unbounded operators, we should consider a pre-quantum classical statistical model based on the Gelfand triple: $H_c^+ \subset H_c \subset H_c^-$.

[7] We emphasize that one can consider epistemic models without any relation to ontic models. Moreover, one can even assume (as it is commonly done in quantum mechanics) that an underlying ontic model cannot be constructed even in principle. However, we stay on the realist position and suppose that any epistemic model is induced by some ontic model.

One can also postulate that (see, e.g., [4] pp 301–5)

Postulate SS. *Each quantum statistical state* $D \in \mathcal{D}$ *corresponds to a classical statistical state* $\rho \in S$.

Thus there is given a map

$$T : S(\Omega) \to \mathcal{D}(H_c). \tag{4}$$

Moreover, the following is often postulated (see, e.g., the theorem of Kochen and Specker [14]; in von Neumann book [4] it can be derived from equality (**Dis**$_2$), p 313):

Postulate F. *Let* $\phi : \mathbf{R} \to \mathbf{R}$ *be a Borel function such that, for any variable* $f \in V$, $\phi(f) \in V$. *Then* $T(\phi(f)) = \phi(T(f))$.

Both models under consideration—a classical model (which we are looking for) and the quantum model N_{quant}—are statistical; the final outputs of both models are averages: $\langle f \rangle_\rho$ and $\langle A \rangle_D$, which are defined by (1) and (2), respectively. One could postulate (see, e.g., [4], p 301) that

Postulate AVC. *Classical and quantum averages coincide.*

In such a case one has

$$\langle f \rangle_\rho = \langle A \rangle_D, \qquad A = T(f), \qquad D = T(\rho). \tag{5}$$

Thus

$$\int_\Omega f(\omega)\, \mathrm{d}\rho(\omega) = \operatorname{Tr} DA, \qquad A = T(f), \qquad D = T(\rho). \tag{6}$$

As was mentioned, these postulates were considered, in particular, by von Neumann. Finally, he also postulated that

Postulate AD. *The correspondence map T is additive*,

$$T(f_1 + \cdots + f_n + \cdots) = T(f_1) + \cdots + T(f_n) + \cdots, \tag{7}$$

for any sequence of variables $f_1, \ldots, f_n, \ldots \in V(\Omega)$.[8]

Already in the 1930s von Neumann demonstrated that a correspondence map T satisfying postulates VO, SS, F, AVC, AD does not exist [4]. Bell [12] paid attention to the fact that not all von Neumann's postulates were physically justified. He (and not only he, see Ballentine [15, 29] for details) strongly criticized postulate AD as totally non-physical [12]. Bell also strongly criticized postulate VO. He pointed out that it might happen that a few different physical variables are mapped into the same physical observable. He proposed to eliminate postulates VO, AD and even consider, instead of the postulate F, a weaker condition:

Postulate RVC. *Ranges of values of a variable* $f \in V$ *and the corresponding quantum observable* $A = T(f)$ *coincide.*

Then he proved [12] that there is still no such correspondence map T. Nevertheless, let us suppose that a pre-quantum ontic model exists. It is natural to ask the following questions:

'Which postulate does block the construction of the correspondence map T? Which postulate is really non-physical?'

[8] It is important to remark that von Neumann did not assume that observables $T(f_1), \ldots, T(f_n), \ldots$ could be measured simultaneously!

2.5. On correspondence between ranges of values of classical variables and quantum observables

We emphasize that physical variables $f \in V$ and observables $F \in O$ are defined on different sets of parameters and therefore they could have different ranges of values, see [54] for a detailed analysis of this problem. In general, a measurement process induces some loss of information about the (ontic) state ω.[9] Therefore an observable is only an approximation of a physical variable. It seems that postulate RVC is non-physical (and consequently its stronger form—postulate F).

We shall show that it is possible to construct a very natural (from the physical viewpoint) classical statistical model such that it is mapped onto the quantum model with a map T which satisfies postulates VO, SS, AVC and even postulate AD (which has been so often criticized). We pay attention to the fact that if the space of physical variables $V(\Omega)$ is an $\mathbf{R}$-linear space then postulate AD can be written as

Postulate RL. *The correspondence map* $T : V(\Omega) \to \mathcal{L}_s(H_c)$ *is* $\mathbf{R}$*-linear.*

Thus we solved the mathematical problem of constructing a pre-quantum classical statistical model and the correspondence map T having natural properties, see section 3. This is the end of the very long mathematical story about the existence of a pre-quantum classical statistical model. But the physical analysis of the problem of correspondence between classical statistical mechanics and quantum mechanics should be continued.

2.6. The role of the Planck constant in classical → quantum correspondence

Our analysis demonstrated, see sections 5 and 6, that the mathematical construction based on the precise equality of classical and quantum averages, see postulate AVC, (5), (6), looks rather restrictive in the physical framework. We proceed with a new classical statistical model having a larger class of physical variables. Here the fundamental equality (5) is violated, but it seems that this corresponds to the real physical situation. Postulate VO is also violated (the map T is still onto $\mathcal{L}_s$, but it is not one-to-one anymore). The crucial point is that, instead of (5), in a new model we have only

$$\langle f \rangle_\rho = \langle T(f) \rangle_{T(\rho)} + o(h), \qquad h \to 0, \tag{8}$$

where h is the Planck constant[10]. In mathematical models, this equality has the form

$$\int_\Omega f(\omega)\,\mathrm{d}\rho(\omega) = \mathrm{Tr}\, DA + o(h), \qquad A = T(f), \qquad D = T(\rho). \tag{9}$$

We claim that

The quantum statistical theory is the result of neglecting $o(h)$-terms in classical (pre-quantum) averages.

Considering the quantum model as the $\lim_{h\to 0}$ of a classical model is a rather unusual viewpoint to the relation between 'classical' and 'quantum'. It is really inverse to the conventional viewpoint (which is presented mathematically in so-called deformation quantization, see, e.g., [47] and [48–50]). In the conventional approach, the classical phase space mechanics can be obtained as $\lim_{h\to 0}$ of the quantum mechanics. To escape misunderstanding, we should explain this point in more detail. One should sharply distinguish two classical statistical models:

[9] In fact, quantum measurements induce huge loss of information in the process of extracting information about properties of microscopic structures with the aid of macroscopic measurement devices.

[10] In mathematical considerations h is not a constant, but a small parameter and $o(h)$ is defined by $\lim_{h\to 0} \frac{o(h)}{h} = 0$.

(1) Classical statistical mechanics on the 'classical phase space' $\Omega = \mathbf{R}^3 \times \mathbf{R}^3$.
(2) The pre-quantum classical statistical model.

In our approach, the latter model is based on an infinite-dimensional phase space Ω.

To distinguish statistical and dynamical problems, in this paper we shall consider the case of the real Hilbert space H. Thus in all the above considerations the complex Hilbert space H_c should be changed to the real Hilbert space H. In particular, $\mathcal{L}_s \equiv \mathcal{L}_s(H), D \equiv \mathcal{D}(H)$, and so on. The case of the complex Hilbert state space will be considered in the next paper.

3. Gaussian measures on Hilbert spaces

Let H be a real Hilbert space and let $A : H \to H$ be a continuous self-adjoint linear operator. The basic mathematical formula which will be used in this paper is the formula for a Gaussian integral of a quadratic form

$$f(x) \equiv f_A(x) = (Ax, x). \tag{10}$$

Let $\mathrm{d}\rho(x)$ be a σ-additive Gaussian measure on the σ field F of Borel subsets of H. This measure is determined by its covariation operator $B : H \to H$ and mean value $m \equiv m_\rho \in H$. For example, B and m determine the Fourier transform of ρ:

$$\tilde{\rho}(y) = \int_H \mathrm{e}^{\mathrm{i}(y,x)}\,\mathrm{d}\rho(x) = \mathrm{e}^{\frac{1}{2}(By,y)+\mathrm{i}(m,y)}, \qquad y \in H.$$

In what follows, we restrict our considerations to Gaussian measures with zero mean value $m = 0$, where

$$(m, y) = \int_H (y, x)\,\mathrm{d}\rho(x) = 0$$

for any $y \in H$. Sometimes the symbol ρ_B will be used to denote the Gaussian measure with the covariation operator B and $m = 0$. We recall that the covariation operator $B \equiv \operatorname{cov}\rho$ is defined by

$$(By_1, y_2) = \int (y_1, x)(y_2, x)\,\mathrm{d}\rho(x), \qquad y_1, y_2 \in H, \tag{11}$$

and has the following properties:

(a) $B \geqslant 0$, i.e., $(By, y) \geqslant 0$, $y \in H$.
(b) B is a self-adjoint operator, $B \in \mathcal{L}_s(H)$.
(c) B is a trace-class operator and

$$\operatorname{Tr} B = \int_H \|x\|^2\,\mathrm{d}\rho(x). \tag{12}$$

The right-hand side of (12) defines *dispersion* of the probability ρ. Thus for a Gaussian probability we have

$$\sigma^2(\rho) = \operatorname{Tr} B. \tag{13}$$

We pay attention to the fact that the list of properties of the covariation operator of a Gaussian measure differs from the list of properties of a von Neumann density operator only by one condition: $\operatorname{Tr} D = 1$, for a density operator D.

By using (11) we can easily find the Gaussian integral of the quadratic form $f_A(x)$ defined by (10),

$$\int_H f_A(x)\,\mathrm{d}\rho(x) = \int_H (Ax, x)\,\mathrm{d}\rho(x) = \sum_{i,j=1}^{\infty} (Ae_i, e_j) \int_H (e_i, x)(e_j, x)\,\mathrm{d}\rho(x)$$
$$= \sum_{i,j=1}^{\infty} (Ae_i, e_j)(Be_i, e_j),$$

where $\{e_i\}$ is some orthonormal basis in H. Thus

$$\int_H f_A(x)\,\mathrm{d}\rho(x) = \mathrm{Tr}\, BA. \tag{14}$$

We have presented some facts about Gaussian measures on Hilbert space; there many books where one can find detailed presentation; I would like to recommend an excellent short book of A V Skorohod [55], see also [56–59] for applications to mathematical physics.

4. The classical statistical model on the real Hilbert space inducing the quantum model

4.1. The classical model with Gaussian statistical states

Let us consider classical physical systems which have H as the state space, so $\Omega = H$. Ensembles of such systems are described by probability measures (*statistical states*) on the σ field of Borel subsets F. We consider a classical statistical model such that the class of statistical states consists of Gaussian measures ρ on H having zero mean value, $m_\rho = 0$, and unit dispersion

$$\sigma^2(\rho) = \int_H \|x\|^2\,\mathrm{d}\rho(x) = 1.$$

These are Gaussian measures having covariance operators with the unit trace, see equality (13). Denote the class of such probabilities by the symbol $S_G \equiv S_G(H)$. In our model, the class of physical variables consists of quadratic forms $f_A(x)$, see (10). We denote this class by $V_{\mathrm{quad}} \equiv V_{\mathrm{quad}}(H)$. We remark that this is a linear space (over $\mathbf{R}$). We consider the following classical statistical model:

$$M_{\mathrm{quad}} = (S_G(H), V_{\mathrm{quad}}(H)).$$

As always in a statistical model, we are interested only in averages of physical variables $f \in V_{\mathrm{quad}}(H)$ with respect to statistical states $\rho \in S_G(H)$. Here we have

$$\langle f \rangle_\rho = \int_H f(x)\,\mathrm{d}\rho(x). \tag{15}$$

We emphasize that by (14)

$$\langle f \rangle_\rho = \mathrm{Tr}\, BA \qquad \text{for} \quad f \equiv f_A. \tag{16}$$

4.2. The correspondence map

Let us consider the following map T from the classical statistical model $M_{\mathrm{quad}} = (S_G(H), V_{\mathrm{quad}}(H))$ to the quantum statistical model $N_{\mathrm{quant}} = (\mathcal{D}(H), \mathcal{L}_{\mathrm{s}}(H))$:

$$T : S_G(H) \to \mathcal{D}(H), \qquad T(\rho) = \mathrm{cov}\,\rho \tag{17}$$

(the Gaussian measure ρ_B is represented by the density matrix D which is equal to the covariation operator of this measure), and we define

$$T : V_{\mathrm{quad}}(H) \to \mathcal{L}_{\mathrm{s}}(H), \qquad T(f) = \tfrac{1}{2} f''(0) \tag{18}$$

(thus a variable $f \in V_{\mathrm{quad}}(H)$ is represented by its second derivative). In principle, one could choose normalization constants in an arbitrary way: $T(\rho) = \alpha \operatorname{cov} \rho, T(f) = \beta f''(0)$, $\alpha\beta = 1/2$ (at least in a purely mathematical considerations)[11].

4.3. Differentiable and analytic functions

The differential calculus for maps $f : H \to \mathbf{R}$ does not differ so much from the differential calculus in the finite-dimensional case, $f : \mathbf{R}^n \to \mathbf{R}$. Instead of the norm on $\mathbf{R}^n$, one should use the norm on H. We consider the so-called Frechet differentiability [56]. Here a function f is differentiable if it can be represented as

$$f(x_0 + \Delta x) = f(x_0) + f'(x_0)(\Delta x) + o(\Delta x), \qquad \text{where} \quad \lim_{\|\Delta x\| \to 0} \frac{\|o(\Delta x)\|}{\|\Delta x\|} = 0.$$

Here the derivative $f'(x)$ is a continuous linear functional on H; so it can be identified with the element $f'(x) \in H$. Then we can define the second derivative as the derivative of the map $x \to f'(x)$ and so on. A map f is differentiable n-times iff (see, e.g., [56])

$$f(x_0 + \Delta x) = f(x_0) + f'(x_0)(\Delta x) + \frac{1}{2} f''(x_0)(\Delta x, \Delta x) + \cdots + \frac{1}{n!} f^{(n)}(x_0)(\Delta x, \ldots, \Delta x) + o_n(\Delta x), \tag{19}$$

where $f^{(n)}(x_0)$ is a symmetric continuous n-linear form on H and

$$\lim_{\|\Delta x\| \to 0} \frac{\|o_n(\Delta x)\|}{\|\Delta x\|^n} = 0.$$

For us it is important that $f''(x_0)$ can be represented by a symmetric operator

$$f''(x_0)(u, v) = (f''(x_0)u, v), \qquad u, v \in H$$

(this fact is well known in the finite-dimensional case: the matrix representing the second derivative of any two times differentiable function $f : \mathbf{R}^n \to \mathbf{R}$ is symmetric). In (18) this operator was considered for the particular choice $x_0 = 0$. We remark that in this case

$$f(x) = f(0) + f'(0)(x) + \frac{1}{2} f''(0)(x, x) + \cdots + \frac{1}{n!} f^{(n)}(0)(x, \ldots, x) + o_n(x). \tag{20}$$

We recall that a function $f : H \to \mathbf{R}$ is (real) analytic if it can be expanded into series

$$f(x) = f(0) + f'(0)(x) + \frac{1}{2} f''(0)(x, x) + \cdots + \frac{1}{n!} f^{(n)}(0)(x, \ldots, x) + \cdots \tag{21}$$

which converges uniformly on any ball of H, see [60] for details.

4.4. The fundamental theorem on classical $\to$ quantum correspondence

Theorem 4.1. *The map T defined by (17), (18) is one-to-one on the spaces of statistical states and physical variables, $S_G(H)$ and $V_{\mathrm{quad}}(H)$); the map $T : V_{\mathrm{quad}}(H) \to \mathcal{L}_{\mathrm{s}}(H)$ is linear and the equality of classical and quantum averages (6) holds.*

[11] We shall see that in fact $2 = 2!$ is just the normalization coefficient for the second term in the Taylor expansion.

Proof. The map (17) is one-to-one, because a Gaussian measure ρ (with $m_\rho = 0$) is uniquely defined by its covariation operator. The map $T : V_{\mathrm{quad}}(H) \to \mathcal{L}_{\mathrm{s}}(H)$ is one-to-one, since the quadratic form $x \to f(x)$ is uniquely determined by its second derivative. It is linear, because the operation of differentiation is linear. The fundamental equality (6) is a consequence of the equality (14) for Gaussian measures:

$$\int_H f(x)\,\mathrm{d}\rho(x) = \frac{1}{2}\int_H (f''(0)x, x)\,\mathrm{d}\rho(x) = \frac{1}{2}\,\mathrm{Tr}\,Bf''(0). \tag{22}$$

□

Thus von Neumann postulates VO, SS, AVC hold as well as postulate RL. The latter implies that even postulate AD holds for a finite number of physical variables $f_1, \ldots, f_n \in V_{\mathrm{quad}}(H)$. We recall that von Neumann used this postulate for an infinite number of physical variables (this is crucial in his 'no-go' considerations). We consider the mathematical formulation of postulate AD for infinitely many variables in the appendix.

5. The observation process and loss of information

5.1. Measurements on systems with the infinite number of degrees of freedom

Our theory describes the following physical situation. There is an infinite-dimensional space of classical states $\Omega = H$; in principle, it can be interpreted as the *space of classical fields*. Statistical states are represented by a special class of Gaussian distributions on the space of fields. Physical variables are quadratic forms of fields. Physical variables depend on the infinite number of degrees of freedom:

$$f = f(x_1, \ldots, x_n, \ldots)$$

We measure such quantities by using some macroscopic measurement apparatuses. We could not even in principle extract information about an infinite number of variables. There is no hope to reproduce exactly the ontic quantity. For example, huge disturbances can be produced by macroscopic measurement devices. But I am not sure that it is really the point. It seems the point is that a measurement which is performed during a finite interval of time could not give us complete information about an infinite number of variables determining the ontic physical variable. In particular, there is no reason to expect that the range of values of a variable $f(x)$ would coincide with the range of values of the corresponding observable $T(f)$. In our approach, postulate RVC (and its stronger form—postulate F) are not physical. On the other hand, the von Neumann postulate AD (which was so strongly criticized by many authors, see the introduction) does not induce any problem. The situation with the von Neumann postulate VO is more complicated. As we have seen, theorem 4.1, it is possible to create a pre-quantum model in that this postulate is not violated. However, in a more general (and natural from the physical viewpoint) model, see section 6, this postulate is violated.

The only thing that one can expect from an adequate observational model is coincidence (in fact, only with some precision, but see section 6 for details) of ontic and observational averages. And we have this in our approach; see theorem 4.1, equality (22).

The classical statistical model $M_{\mathrm{quad}} = (S_G, V_{\mathrm{quad}})$ naturally induces the statistical observational model $N_{\mathrm{quant}} = (\mathcal{D}, \mathcal{L}_{\mathrm{s}})$. As was already pointed out, this construction can be considered as the end of the long story about the possibility of finding a pre-quantum classical statistical model. It was commonly believed that such a pre-quantum model does not exist at all. Nevertheless, we demonstrated that it exists. This induces new interesting questions:

(a) Why does the space of classical statistical states consist of Gaussian measures?
(b) Why does the space of classical physical variables consist of quadratic forms?
(c) What is the role of the Planck constant h in our approach?

5.2. The statistical origin of Gaussian pre-quantum states

The choice of Gaussian probability distributions as statistical states is natural from the probabilistic viewpoint. By the central limit theorem (which is also valid for H-valued random variables, see [61]) a Gaussian probability distribution appears as the integral effect of infinitely many independent random influences. Of course, it is important that in our case each random influence is given by a random variable $\xi(\omega) \in H$. Thus we consider the infinite number of degrees of freedom. A Gaussian distribution ρ is the integral result of influences of infinitely many such ξ. But from the purely measure-theoretical viewpoint there is not so much difference between the origin of Gaussian probability distribution on H and $\mathbf{R}^n$.

The explanation of consideration of quadratic functions on the state space is a complicated problem which will be discussed in more detail in section 6.[12]

5.3. On the role of the Planck constant

Another problem is the absence of the quantum h in our framework. Many authors would not consider this as a problem. It is often assumed that $h = 1$. However, the Planck constant h (considered as a small parameter) is extremely important in so-called *deformation quantization*, see e.g. [47–50] and, in particular, in establishing the correspondence principle between quantum and classical models [47–50]. The ordinary classical mechanics on the phase space $\mathbf{R}^3 \times \mathbf{R}^3$ is obtained as the limit $h \to 0$ of quantum mechanics formulated with the aid of the calculus of pseudo-differential operators. The ideas of deformation quantization are important for us. However, they will be used in a very perverse form: *We shall see that the quantum observational model* $N_{\text{quant}} = (\mathcal{D}, \mathcal{L}_{\text{s}})$ *is* $\lim_{h\to 0}$ *of a classical statistical model on the space of microstates H (the new model will extend the model* $M_{\text{quad}} = (S_G, V_{\text{quad}})$*).*

5.4. Second quantization

Finally, we emphasize again that in fact there are two classical statistical models: ordinary classical statistical mechanics (CSM) on the phase space $\mathbf{R}^3 \times \mathbf{R}^3$ and classical statistical mechanics on the infinite-dimensional Hilbert space H. It is well known that the latter classical mechanics can be quantized again. This is the procedure of second quantization. This procedure gives nothing else than operator quantization approach to QFT; see, e.g., [62]. There the principle of correspondence between classical and quantum models for systems with the infinite number of degrees of freedom can also be established. The easiest way do to this is to repeat Weyl's considerations and use the calculus of infinite-dimensional pseudo-differential operators (PDO). Such a calculus was developed on the physical level of rigorousness in [62] and on the mathematical level of rigorousness by Smolyanov and the author [63, 60]; finally, the principle of correspondence was proved [64]. But in this paper we are not interested in QFT. We only remark that methods developed in this paper can be generalized to QFT which can also be presented as the T projection of a classical statistical model.

[12] Of course, we can present a simple, but very important motivation. Such a choice of the space of classical physical variables is justified because it works well and induces the von Neumann trace rule for averages on the level of the observational model.

6. To the quantum model through neglecting $o(h)$-terms in the classical model with infinite-dimensional state space

6.1. The classical model with analytic physical variables

Let us consider another (more physically justified) classical statistical model in that $\Omega = H$, but the class of statistical states consists of Gaussian measures with zero mean value and dispersion

$$\sigma^2(\rho) = \int_H \|x\|^2 \, \mathrm{d}\rho(x) = h, \tag{23}$$

where $h > 0$ is a small real parameter. Denote such a class by the symbol $S_G^h(H)$. For $\rho \in S_G^h(H)$, we have

$$\operatorname{Tr} \operatorname{cov} \rho = h. \tag{24}$$

Let $h > 0$ be a constant. We have for any Gaussian measure ρ_B

$$\langle f \rangle_{\rho_B} = \int_H f(x) \, \mathrm{d}\rho_B(x) = \int_H f(\sqrt{h} y) \, \mathrm{d}\rho_D(y).$$

We did the change of variables (scaling),

$$y = \frac{x}{\sqrt{h}}. \tag{25}$$

We remark that any linear transformation (in particular, scaling) preserves the class of Gaussian measures. To find the covariation operator D of a new Gaussian measure ρ_D, we compute its Fourier transform:

$$\tilde{\rho}_D(\xi) = \int_H \mathrm{e}^{\mathrm{i}(\xi, y)} \, \mathrm{d}\rho_D(y) = \int_H \mathrm{e}^{\mathrm{i}(\xi, \frac{x}{\sqrt{h}})} \, \mathrm{d}\rho_B(x) = \mathrm{e}^{-\frac{1}{2h}(B\xi, \xi)}.$$

Thus

$$D = \frac{B}{h} = \frac{\operatorname{cov} \rho}{h}. \tag{26}$$

We shall use this formula later.

Let us consider a functional space $\mathcal{V}(H)$ which consists of analytic functions of exponential growth preserving the state of vacuum:

$$f(0) = 0 \text{ and there exist } C, \alpha \geqslant 0 : |f(x)| \leqslant C \, \mathrm{e}^{\alpha \|x\|}.$$

We remark that any function $f \in \mathcal{V}(H)$ is integrable with respect to any Gaussian measure on H; see, e.g., [55, 56]. Let us consider the classical statistical model

$$M_a^h = \big(S_G^h(H), \mathcal{V}(H)\big).$$

6.2. Asymptotic expansion of classical averages

Let us find the average of a variable $f \in \mathcal{V}(H)$ with respect to a statistical state $\rho_B \in S_G^h(H)$:

$$\langle f \rangle_{\rho_B} = \int_H f(x) \, \mathrm{d}\rho_B(x) = \int_H f(\sqrt{h} y) \, \mathrm{d}\rho_D(y) = \sum_{n=2}^{\infty} \frac{h^{n/2}}{n!} \int_H f^{(n)}(0)(y, \ldots, y) \, \mathrm{d}\rho_D(y), \tag{27}$$

where the covariation operator D is given by (26). We remark that

$$\int_H (f'(0), y) \, \mathrm{d}\rho(y) = 0,$$

because the mean value of ρ is equal to zero. Since $\rho_B \in S_G^h(H)$, we have

$$\mathrm{Tr}\, D = 1. \tag{28}$$

The change of variables in (27) can be considered as rescaling of the magnitude of statistical (Gaussian) fluctuations. Fluctuations which were considered as very small,

$$\sigma(\rho) = \sqrt{h}, \tag{29}$$

(where h is a small parameter) are considered in the new scale as standard normal fluctuations. By (27) we have

$$\langle f \rangle_\rho = \frac{h}{2} \int_H (f''(0)y, y)\, \mathrm{d}\rho_D(y) + o(h), \qquad h \to 0, \tag{30}$$

or

$$\langle f \rangle_\rho = \frac{h}{2} \mathrm{Tr}\, D f''(0) + o(h), \qquad h \to 0. \tag{31}$$

We see that the classical average (computed in the model $M_a^h = (S_G^h(H), \mathcal{V}(H))$ by using measure-theoretic approach) is approximately equal to the quantum average (computed in the model $N_{\mathrm{quant}} = (\mathcal{D}(H), \mathcal{L}_{\mathrm{s}}(H))$ by the von Neumann trace formula).

6.3. Classical → quantum correspondence

The equality (31) can be used as the motivation for defining the following classical → quantum map T from the classical statistical model $M_a^h = \left(S_G^h, \mathcal{V}\right)$ to the quantum statistical model $N_{\mathrm{quant}} = (\mathcal{D}, \mathcal{L}_{\mathrm{s}})$,

$$T : S_G^h(H) \to \mathcal{D}(H), \qquad D = T(\rho) = \frac{\mathrm{cov}\, \rho}{h} \tag{32}$$

(the Gaussian measure ρ is represented by the density matrix D which is equal to the covariation operator of this measure normalized by the Planck constant h):

$$T : \mathcal{V}(H) \to \mathcal{L}_{\mathrm{s}}(H), \qquad A_{\mathrm{quant}} = T(f) = \frac{h}{2} f''(0). \tag{33}$$

Our previous considerations can be presented as

Theorem 6.1. *The map T defined by (32) and (33) is one-to-one on the space of statistical states $S_G^h(H)$; the map $T : \mathcal{V}(H) \to \mathcal{L}_{\mathrm{s}}(H)$ is linear and the classical and quantum averages are asymptotically, $h \to 0$, equal; see (31).*

However, contrary to the model $M_{\mathrm{quad}} = (S_G(H), V_{\mathrm{quad}}(H))$, the correspondence between physical variables $f \in \mathcal{V}(H)$ and physical observables $A \in \mathcal{L}_{\mathrm{s}}(H)$ is not one-to-one[13].

[13] A large class of physical variables is mapped into one physical observable. We can say that the quantum observational model N_{quant} does not distinguish physical variables of the classical statistical model M_a^h. The space $\mathcal{V}(H)$ is split into equivalence classes of physical variables: $f \sim g \leftrightarrow f''(0) = g''(0)$. Each equivalence class W is characterized by a continuous self-adjoint operator $A_{\mathrm{quant}} = \frac{h}{2} f''(0)$, where f is a representative of physical variables from the class W. The restriction of the map T on the space of quadratic observables $V_{\mathrm{quad}}(H)$ is one-to-one. Of course, the set of variables $\mathcal{V}(H)$ can be essentially extended (in particular, we can consider smooth functions on the Hilbert space, instead of analytic functions). However, we emphasize that such an extension would have no effect to the quantum observational model.

6.4. Physical conclusions

Our approach is based on considering the Planck constant h as a small parameter. Let us consider some classical statistical model $M = (S, V)$ and an observational model $N = (D, O)$. Suppose that in this observational model quantities of the magnitude $\ll h$ could not be observed. Therefore, for an observable $A \in O$, its average can be calculated only up to the magnitude h. On the other hand, in the classical statistical model M average contains even terms of the magnitude $o(h)$. Such terms are neglected in the correspondence between M and N. We just formalized this procedure by considering mathematical models $M_a^h = \left(S_G^h, \mathcal{V}\right)$ and $N_{\text{quant}} = (\mathcal{D}, \mathcal{L}_{\text{s}})$.

In the observational model, we neglect terms of the magnitude $o(h)$ in all statistical averages. This is our *understanding of quantization*.

6.4.1. Conclusion. Quantum mechanics is an approximative statistical description of nature based on extracting quantities of the magnitude h and neglecting quantities of the magnitude $o(h)$.

6.5. On the statistical meaning of the Planck constant

The approach based on the scaling of statistical states has some interesting physical consequences. The space $S_G^h(H)$ of statistical states of the pre-quantum classical model consists of Gaussian distributions with zero mean value and dispersion of the magnitude h. If h is very small, then such a ρ is concentrated in a very small neighbourhood of the state $\omega = 0$. Let us interpret $\omega = 0$ as the *state of vacuum*[14]. Thus von Neumann density matrices represent Gaussian statistical states (on the infinite-dimensional state space H) which are very narrow concentrated around the vacuum state $\omega = 0$. Such states can be considered as *fluctuations of vacuum*, cf [18, 51, 52]. Therefore in our approach the Planck constant has only statistical meaning—dispersion of fluctuations of vacuum.

Such a statistical viewpoint to the small parameter gives the possibility of applying the quantum formalism in any statistical model (in any domain of science) which contains statistical states having dispersion of the magnitude κ, where κ is some small parameter. It is clear that such a model describes very fine effects. In a coarser approximation, such statistical states would be considered as states with zero dispersion.

Finally, we pay attention to the (at least theoretic) possibility of constructing finer quantum observational models for the pre-quantum classical statistical model by using some small parameter

$$\kappa \ll h. \tag{34}$$

7. Gaussian measures inducing pure quantum states: statistical meaning of the wavefunction

We start with statistical interpretation of pure quantum states.

7.1. Gaussian underground

In QM a pure quantum state is given by a normalized vector $\psi \in H : \|\psi\| = 1$. The corresponding statistical state is represented by the density operator:

$$D_\psi = \psi \otimes \psi. \tag{35}$$

[14] We remark that this is the classical vacuum field and not a vacuum state of QFT.

In particular, the von Neumann's trace formula for expectation has the form

$$\mathrm{Tr}\, D_\psi A = (A\psi, \psi). \tag{36}$$

Let us consider the correspondence map T for statistical states for the classical statistical model $M_a^h = \left(S_G^h, \mathcal{V}\right)$, see (32). It is evident that the pure quantum state ψ (i.e., the state with the density operator D_ψ) is the image of the Gaussian statistical mixture ρ_ψ of states $\omega \in H$. Here the measure ρ_ψ has the covariation operator

$$B_\psi = h D_\psi. \tag{37}$$

Thus

$$(B_\psi y_1, y_2) = \int_H (y_1, x)(y_2, x)\, \mathrm{d}\rho\psi(x) = h(y_1, \psi)(\psi, y_2).$$

This implies that the Fourier transform of the measure ρ_ψ has the form

$$\tilde{\rho}_\psi(y) = \mathrm{e}^{-\frac{h}{2}(y,\psi)^2}, \qquad y \in H.$$

This means that the measure ρ_ψ is concentrated on the one-dimensional subspace

$$H_\psi = \{x \in H : x = s\psi, s \in \mathbf{R}\}.$$

This is one-dimensional Gaussian distribution. It is very important to pay attention to the following trivial mathematical fact:

Concentration on the one-dimensional subspace H_ψ does not imply that the Gaussian measure ρ_ψ is a pure state of the Dirac-type δ-function on the classical state space $\Omega = H$.

7.2. *Ontic states and wavefunctions*

In our ontic model, states are represented by vectors of the Hilbert space H. Since pure states in QM are also represented by vectors of H, one might try to identify them. The important difference is that any vector belonging to H represents an ontic state, but only normalized vectors of H represent pure quantum states. However, this is not the crucial point. The crucial point is that the von Neumann density operator $D_\psi = \psi \otimes \psi$ has nothing to do with the ontic state ψ, even in the case of $\|\psi\| = 1$. The density operator describes not an individual state, but a Gaussian statistical ensemble of individual states. States in this ensemble can have (with corresponding probabilities) any magnitude.

7.2.1. Conclusion. Quantum pure states $\psi \in H$, $\|\psi\| = 1$, represent Gaussian statistical mixtures of classical states. Therefore, quantum randomness is ordinary Gaussian randomness (so it is reducible to the classical ensemble randomness).

Dispersion of the Gaussian measure ρ_ψ has the magnitude of the Planck constant h. Thus ρ_ψ is very narrow Gaussian distribution concentrated around the vacuum state $\omega = 0$. Roughly speaking, the whole quantum theory is about fluctuations of vacuum. But we recall that this is a very approximative theory, because only terms of the magnitude h are taken into account.

8. Incompleteness of quantum mechanics

Assume that our classical statistical model provides the adequate description of physical reality. This would imply that *quantum mechanics is not complete*—since it does not describe 'individual states' $\omega \in \Omega$. However, it seems that it is practically impossible to verify this prediction experimentally, because it is impossible to prepare 'pure ontic states' ω for microscopic systems. It is easier to prove that quantum mechanics is not complete even as a statistical model, namely that in nature there exist classical statistical states (different from δ_ω states) which have no image in the quantum model.

Let us start with 'pure non-quantum states'. Let $\psi \in H$, but its norm need not be equal to 1. Let us consider the corresponding Gaussian statistical state ρ_ψ; see (37). This state represents the Gaussian distribution concentrated on the real line. We pay attention to the fact that by scaling the vector ψ we obtain a completely different Gaussian distribution. The only commonality between measures ρ_ψ and $\rho_{\lambda\psi}$, $\lambda \in \mathbf{R}$, is that they are concentrated on the same real line. But they have different dispersions (and so shapes). In particular, it is impossible to represent all scalings by the normalized vector $\psi/\|\psi\|$.

Suppose now that $\|\psi\| = o(1), h \to 0$. In our mathematical model, there exist classical statistical states ρ_ψ with covariance matrices $B_\psi = h\psi \otimes \psi$, see (37). However, the quantum statistical model N_{quant} does not contain images of such states, because $\sigma^2(\rho) = o(h)$.

In the same way we can consider any classical statistical state ρ having the dispersion $\sigma^2(\rho)$ such that

$$\sigma^2(\rho) = o(h). \tag{38}$$

Thus if we consider a general classical statistical model M_{general} containing all Gaussian states with zero average (without any restriction to the magnitude of dispersion) then it could not be mapped onto QM.

9. Interpretation and comparison with other realistic pre-quantum models

9.1. Ensemble interpretation

As was already pointed out in the introduction, basic elements of ontic reality (i.e., reality independent from observations) are systems with the infinite number of degrees of freedom, say 'classical fields'. Statistical states which we are able to prepare in laboratories and which correspond to statistical states described by quantum mechanics are Gaussian distributions of such fields. The mean value of these Gaussian fluctuations is the vacuum field, $\omega = 0$. Statistical deviations from the vacuum field

$$\sigma(\rho) = \sqrt{\int_{L_2(\mathbf{R}^n, \mathrm{d}x)} \left(\int_{\mathbf{R}^n} |\psi(x)|^2 \, \mathrm{d}x \right) \mathrm{d}\rho(\psi)} = \sqrt{h}$$

(here $H = L_2(\mathbf{R}^n, \mathrm{d}x)$ is the space of square integrable functions with respect to the Lebesgue measure on $\mathbf{R}^n$). Dynamics is given by Gaussian processes $\xi(t, \lambda)$ with values in the Hilbert space; here λ is a chance parameter. Thus if at the moment t_0 an ensemble of fields was described by a random variable $\xi_0(\lambda) \in H$, then at the moment t we obtain an ensemble of fields described by the random variable $\xi(t, \lambda) \in H$. A Gaussian random variable can take any value $\xi \in H$. There is no restriction on the magnitude of $\xi(t, \omega) : \|\xi(t, \lambda)\| \in [0, +\infty)$. In particular, there is no normalization of states by 1. By using the language of stochastic processes we write

$$E\xi(t) = 0, \qquad \sigma(\xi(t)) = \sqrt{h}.$$

We use the ensemble (or statistical) interpretation of quantum states, since they are images, $D = T(\rho)$, of Gaussian statistical states. The only difference from the conventional ensemble (or statistical) interpretation of quantum mechanics (cf Einstein, Margenau, Ballentine [15, 28]) is that we consider ensembles of classical fields, instead of ensembles of particles.

Our approach might be called *pre-quantum classical statistical field theory* (PCSFT).

9.2. Comparison with the views of Schrödinger

Our views are close to the Schrödinger's original views about the wavefunction as a classical scalar field as well as his later ideas to exclude totally particles from quantum mechanics, see [65]. However, the latter program was performed in the QFT framework. In PCSFT we do not consider quantized fields.

9.3. Comparing PCSFT with Bohmian mechanics

The main difference between PCSFT and the Bohmian model is that the Bohmian model still contains particles as real objects. In particular, quantum randomness is due to the randomness of initial states of particles and not randomness of initial states of fields as in PCSFT. But, of course, the presence of a field element, namely the pilot wave, induces some similarities between Bohmian mechanics and PCSFT.

9.4. Comparing PCSFT with stochastic quantum mechanics/SED

The comparison here is very similar to that with Bohmian mechanics: particles are real elements of SED, but not of PCSFT. In SED, quantum randomness is the result of interaction of particles with random media ('fluctuations of vacuum'). In PCSFT particles themselves are images of fluctuating fields. So the crucial point is not the presence of fluctuations of vacuum, but that behind 'quantum particles' there are classical fields. We discuss the interpretation of the background field in PCSFT in the next paragraph.

In PCSFT the zero point field is interpreted in the following way. The real field of vacuum, $\omega = 0$, has zero energy, because for any quadratic form $\mathcal{H}(\omega) = (\mathbf{H}\omega, \omega)$, where $\mathbf{H}$ is a self-adjoint operator, we have $\mathcal{H}(0) = 0$. Nevertheless, we have an analogue of the zero point field in PCSFT. As was pointed out QM is not complete theory, because there exist (at least in the mathematical model) statistical Gaussian states representing fluctuations of the vacuum field with the statistical deviation $\sigma(\rho) = o(\sqrt{h})$. Such statistical states are neglected in the modern observational model, QM, in that only states with $\sigma(\rho) = \sqrt{h}$ are taken into account. But statistical states which we neglect in QM have nonzero average of energy:

$$\langle\mathcal{H}\rangle_\rho = \int_H (\mathbf{H}\omega, \omega)\, \mathrm{d}\rho(\omega).$$

Of course, this average is negligibly small,

$$|\langle\mathcal{H}\rangle_\rho| \leqslant \|\mathbf{H}\| \int_H \|\omega\|^2\, \mathrm{d}\rho(\omega) = \|\mathbf{H}\|\sigma^2(\rho) = o(h)$$

(we considered the case of continuous operator $\mathbf{H} : H \to H$). We can say that PCFT supports the zero point field model.

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Appendix A

A.1. On the von Neumann postulate about correspondence between sums

We consider on the space of self-adjoint operators $\mathcal{L}_s(H)$ the weak topology: $A_n \to A$ iff $(A_n x, y) \to (Ax, y)$ for any $x, y \in H$; and on the functional space $V_{\text{quad}}(H)$ the pointwise convergence. We remark that the space $\mathcal{L}_s(H)$ is sequentially complete in the weak topology. Thus any series $\sum_{n=1}^{\infty} A_n$, $A_n \in \mathcal{L}_s(H)$, which converges in the weak topology determines a continuous operator. Therefore any pointwise convergent series $\sum_{n=1}^{\infty} f_n(x)$, $f_n \in V_{\text{quad}}(H)$, determines a continuous quadratic form:

$$f(x) = \sum_{n=1}^{\infty} f_n(x) = \frac{1}{2}\left(\sum_{n=1}^{\infty} f_n''(0)x, x\right).$$

This equality implies

Proposition A.1. *The correspondence map* $T : V_{\text{quad}}(H) \to \mathcal{L}_s(H)$ *(given by (18)) is continuous; for any pointwise converging series of variables* $f(x) = \sum_{n=1}^{\infty} f_n(x)$, $f_n(x) \in V_{\text{quad}}(H)$ *we have*

$$T\left(\sum_{n=1}^{\infty} f_n\right) = \sum_{n=1}^{\infty} T(f_n).$$

A.2. Finite-dimensional QM as an image of CSM

Let us consider our classical statistical model in the finite-dimensional case. We introduce a new parameter h_{class} which has macroscopic dimension, but it is considered as a small parameter by some 'super-observer'. We consider the classical statistical model

$$M_a^{h_{\text{class}}}(\mathbf{R}^n) = \left(S_G^{h_{\text{class}}}(\mathbf{R}^n), \mathcal{V}(\mathbf{R}^n)\right).$$

This is a special model of classical statistical mechanics (CMS)[15].

Let us now consider a variant of QM in that the state space is finite dimensional. As we consider in this paper only real numbers, we have the model $N_{\text{quant}}(\mathbf{R}^n) = (\mathcal{D}(\mathbf{R}^n), \mathcal{L}_s(\mathbf{R}^n))$. By using the Maclaurin expansion we can establish the T-correspondence between the models $M_a^{h_{\text{class}}}(\mathbf{R}^n)$ and $N_{\text{quant}}(\mathbf{R}^n)$ and obtain the following fundamental equality:

$$\langle f \rangle_\rho = \langle T(f) \rangle_{T(\rho)} + o(h_{\text{class}}), \qquad h_{\text{class}} \to 0. \tag{A.1}$$

A.3. Extension of the space of statistical states

We have seen that the quantum (observational) statistical model can be considered as the image of a classical (ontic) statistical model. In our classical model, the space of statistical states consists of Gaussian distributions having zero mean value and dispersion $\sigma^2(\rho) = h$. Such states describe Gaussian fluctuations of the state of vacuum, $\omega = 0$. The statistical magnitude of fluctuations is equal to h. However, in all our considerations it was important that only the magnitude of fluctuations is *approximately equal to h*. Therefore we can essentially

[15] The conventional model for CSM is given by

$$M = (PM(\mathbf{R}^n), C_b^{\infty}(\mathbf{R}^n)).$$

Here the space of statistical states coincides with the space of all probability measures and the space of physical variables consists of smooth bounded functions.

extend the class of Gaussian classical statistical states and still obtain the same set of quantum states $\mathcal{D}(H)$. Of course, for such a model the correspondence between classical and quantum statistical states would not be one-to-one. Let us consider the space of Gaussian measures on H having zero mean value and dispersion

$$\sigma^2(\rho) = h + o(h), \qquad h \to 0. \tag{A.2}$$

Denote it by the symbol $S_G^{\approx h}(H)$. We consider the following correspondence map between classical and quantum statistical states extending the map (32):

$$T : S_G^{\approx h}(H) \to \mathcal{D}(H), \qquad T(\rho) = \frac{\operatorname{cov} \rho}{\sigma^2(\rho)}. \tag{A.3}$$

We see that the operator $D = T(\rho) \in \mathcal{D}(H)$, so the map T is well defined.

Proposition A.2. *For the map T defined by (A.3) the asymptotic equality of classical and quantum averages (31) holds for any variable $f \in \mathcal{V}(H)$.*

Proof. We have $\langle f\rangle_{\rho_B} = \int_H f(x)\,\mathrm{d}\rho_B(x) = \int_H f(\sigma(\rho_B)y)\,\mathrm{d}\rho_D(y) = \frac{h+o(h)}{2}\int_H (f''(0)y, y)\,\mathrm{d}\rho_D(y) + o(h)$. So we obtained the asymptotic equality (31). $\square$

As was pointed out, two different Gaussian measures $\rho_1, \rho_2 \in S_G^{\approx h}(H)$ can be mapped to the same density operator D. If the condition

$$\sigma^2(\rho_1) - \sigma^2(\rho_2) = o(h), \qquad h \to 0, \tag{A.4}$$

holds, then $T(\rho_1) = T(\rho_2)$.

A.4. Non-Gaussian classical statistical states

Our choice of Gaussian statistical states is based on the central limit theorem for H-valued independent random variables (independent random fluctuations of vacuum). However, in principle, we could not exclude the possibility that in nature there may exist stable non-Gaussian statistical states. We recall that the formula (14) giving the trace expression of integrals of quadratic forms is valid for arbitrary measure μ on H having zero mean value and the finite second moment:

$$\sigma^2(\mu) = \int_H \|x\|^2\,\mathrm{d}\mu(x) < \infty.$$

Denote the set of such probability measures by symbol $PM_2(H)$. Let us consider the classical statistical model

$$M_{a,2}^h = \big(PM_2^h(H),\ \mathcal{V}_2(H)\big),$$

where $PM_2^h(H)$ consists of $\mu \in PM_2(H)$ having the dispersion $\sigma^2(\mu) = h$ and the space of variables $\mathcal{V}_2(H)$ consists of real analytic functions $f : H \to \mathbf{R}$, $f(0) = 0$, having quadratic growth for $x \to \infty$:

$$|f(x)| \leqslant c_1 + c_2\|x\|^2, \qquad x \in H, \quad c_1, c_2 > 0.$$

We find the average of $f \in \mathcal{V}_2(H)$ with respect to $\mu \in PM_2^h(H)$:

$$\begin{aligned}\langle f\rangle_\mu &= \int_H f(x)\,\mathrm{d}\mu(x) = \int_H f(\sigma(\mu)y)\,\mathrm{d}\nu(y)\\ &= \frac{\sigma^2(\mu)}{2}\int_H (f''(0)(y, y)\,\mathrm{d}\nu(x) + \sum_{n=2}^{\infty}\frac{\sigma^2(\mu)}{(2n)!}\int_H f^{(2n)}(0)(y, \dots, y)\,\mathrm{d}\nu(x),\end{aligned}$$

where the measure ν is the scaling of the measure μ induced by the map: $y = \frac{x}{\sigma(\mu)}$. We remark that the covariation operator of the measure ν is obtained as the scaling of the covariation operator of the measure $\mu : D = \operatorname{cov} \nu = \frac{\operatorname{cov} \mu}{\sigma(\mu)}$.

Thus we again have $\langle f \rangle_\mu = \frac{h}{2} \int_H (f''(0)y, y)\, \mathrm{d}\nu(x) + o(h) = \frac{h}{2} \operatorname{Tr} \operatorname{cov} \nu f''(0) + o(h)$.

Hence, the quantum model N_{quant} can be considered as the image of the classical model $M_{a,2}^h$ and classical and quantum averages are equal asymptotically, $h \to 0$. The map T has huge degeneration on the space of statistical states, since a covariation operator does not determine a measure uniquely.

As well as in the Gaussian case, we can consider the space of measures dispersion of which is only approximately equal h:

$$PM_2^{\approx h}(H) = \{\mu \in PM_2(H) : \sigma^2(\mu) = h + o(h), h \to 0\}.$$

The map T can be extended to this class (by increasing degeneration).

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